$Simultaneous\ \ Linear\ Differential\ Equations$

$$x = x(t)$$

$$y = y(t)$$

$$a\,Dx + b\,Dy = F_1(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$c\,Dx + d\,Dy = F_2(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\begin{bmatrix} aD & bD \\ cD & dD \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} F_1(t) & bD \\ F_2(t) & dD \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} aD & F_1(t) \\ cD & F_2(t) \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}}$$

$$Ex.: Solve$$

$$2\frac{dx}{dt}+x+\frac{dy}{dt}=t \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$3\frac{dx}{dt}+2\frac{dy}{dt}-y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solution

$$2Dx + x + Dy = t$$

$$3Dx + 2Dy - y = 0$$

$$(2D+1)x + Dy = t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$3Dx + (2D-1)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\begin{bmatrix} (2D+1) & D \\ 3D & (2D-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} t & D \\ 0 & (2D-1) \end{vmatrix}}{\begin{vmatrix} (2D+1) & D \\ 3D & (2D-1) \end{vmatrix}}$$

$$x = \frac{(2D-1)t-0}{(2D+1)(2D-1)-3D(D)}$$

$$x = \frac{2Dt-t}{4D^2-1-3D^2} = \frac{2-t}{D^2-1}$$

$$(D^2-1)x = 2-t$$

$$D^2-1=0 \Rightarrow D=\pm 1$$

$$x_{\text{homo.}} = c_1 e^{t} + c_2 e^{-t}$$

$$x_p = A + Bt$$

$$x' = B$$

$$x'' = 0$$

$$0 - (A + Bt) = 2 - t$$

$$-A = 2 \quad \Rightarrow A = -2$$

$$-Bt = -t \Rightarrow B = 1$$

$$\therefore x = c_1 e^{t} + c_2 e^{-t} - 2 + t$$

$$y = \frac{\begin{vmatrix} (2D+1) & t \\ 3D & 0 \end{vmatrix}}{\begin{vmatrix} (2D+1) & D \\ 3D & (2D-1) \end{vmatrix}} = \frac{0-3Dt}{D^2-1}$$

$$(D^2-1)y = -3$$

$$D^2-1=0 \;\Rightarrow\; D=\pm 1$$

$$y_{\text{homo.}} = c_3 e^{t} + c_4 e^{-t}$$

$$y_p = A_1$$

$$y' = 0$$

$$y'' = 0$$

$$0 - A_1 = -3 \Rightarrow A_1 = 3$$

$$\therefore y = c_3 e^{t} + c_4 e^{-t} + 3$$

$$\frac{dx}{dt} = Dx = c_1 e^t - c_2 e^{-t} + 1$$

$$\frac{dy}{dt} = Dy = c_3 e^t - c_4 e^{-t}$$

$Subst.$ int$o$ $Eq.$ (2)

$$3(c_1 e^t - c_2 e^{-t} + 1) + 2(c_3 e^t - c_4 e^{-t})$$

$$-(c_3 e^t + c_4 e^{-t} + 3) = 0$$

$$Coeff.\ of\ e^t \Rightarrow 3c_1 + c_3 = 0 \Rightarrow c_3 = -3c_1$$

$$Coeff.\ of\ e^{-t} \Rightarrow -3c_2 - 3c_4 = 0 \Rightarrow c_4 = -c_2$$

$$\therefore x = c_1 e^{t} + c_2 e^{-t} - 2 + t$$

$$y = -3c_1 e^{t} - c_2 e^{-t} + 3$$

$$Ex.:\ Solve$$

$$2\frac{dx}{dt}+\frac{dy}{dt}-x=e^{t}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$3\frac{dx}{dt}+2\frac{dy}{dt}+y=t\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solution:

$$(2D-1)x + Dy = e^t$$

$$3Dx + (2D+1)y = t$$

$$\begin{bmatrix} (2D-1) & D \\ 3D & (2D+1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e^t & D \\ t & (2D+1) \end{vmatrix}}{\begin{vmatrix} (2D-1) & D \\ 3D & (2D+1) \end{vmatrix}}$$

$$x = \frac{(2D+1)e^t - Dt}{(2D-1)(2D+1) - 3D(D)}$$

$$x = \frac{2De^t + e^t - Dt}{4D^2 - 1 - 3D^2} = \frac{3e^t - 1}{D^2 - 1}$$

$$(D^2 - 1)x = 3e^t - 1 \dots\dots\dots\dots\dots (1)$$

$$For \quad y:$$

$$(D^2-1)y=(2D-1)t-3De^t$$

$$=2-t-3e^t$$

$$(D^2-1)y=-3e^t-t+2 \;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$Roots: \pm 1$$

$$x=c_1e^t+c_2e^{-t}+X$$

$$y=c_1e^t+c_2e^{-t}+Y$$

$$X = Ate^t + B$$

$$Ate^t + 2Ae^t - Ate^t - B = 3e^t - 1$$

$$A = \frac{3}{2} \quad and \quad B = 1$$

$$Y = A_o te^t + B_o t + C_o$$

$$A_o te^t + 2A_o e^t - A_o te^t - B_o t - C_o = -3e^t - t + 2$$

$$A_o = -\frac{3}{2}, \quad B_o = 1 \ and \quad C_o = -2$$

$$\therefore\ x = c_1 e^{t} + c_2 e^{-t} + \frac{3}{2} t e^{t} + 1$$

$$y = c_3 e^{t} + c_4 e^{-t} - \frac{3}{2} t e^{t} + t - 2$$

To reduce the No. of constants Subst. x, y and their derivatives in the first equation, (or in second)

$$2\frac{dx}{dt}+\frac{dy}{dt}-x=e^{t} \quad \text{..........................(1)}$$

$$2c_1e^{t}-2c_2e^{-t}+3te^{t}+3e^{t}+$$

$$c_3e^{t}-c_4e^{-t}-\frac{3}{2}te^{t}-\frac{3}{2}e^{t}+1$$

$$-c_1e^{t}-c_2e^{-t}-\frac{3}{2}te^{t}-1=e^{t}$$

$$Coeff.\ of\ \ e^{t}:$$

$$2c_1+c_3-c_1+\frac{3}{2}=1\ or\ c_1+c_3+\frac{1}{2}=0$$

$$or\ c_3=-\frac{1}{2}-c_1$$

$$Coeff.\ of\ \ e^{-t}:$$

$$-2c_2+c_4-c_2=0\ or\ -c_2-c_4=0$$

$$or\ c_4=-3c_1$$

$$\therefore x = c_1 e^{t} + c_2 e^{-t} + \frac{3}{2} t e^{t} + 1$$

$$y = -c_1 e^{t} - 3c_2 e^{-t} - \frac{3}{2} t e^{t} - \frac{1}{2} e^{t} + t - 2$$

$$Ex.:\ Solve$$

$$(2D^{2}+3D-9)x+(D^{2}+7D-14)=4 \quad \ldots\ldots\ldots\ldots(1)$$

$$(D+1)x+(D+2)y=-8\exp(2t) \quad \ldots\ldots\ldots\ldots(2)$$

.

Solution

$$\begin{vmatrix} 2D^2+3D-9 & D^2+7D-14 \\ D+1 & D+2 \end{vmatrix} x = \begin{vmatrix} 4 & D^2+7D-14 \\ -8e^{2t} & D+2 \end{vmatrix}$$

$$[(2D^3+3D^2-9D+4D^2+6D-18)-$$

$$(D^3+7D^2-14D+D^2-7D-14)]x =$$

$$(D+2)4-(D^2+7D-14)(-8e^{2t})$$

$$(D^3-D^2+4D-4)x = 8+32e^{2t}$$

$$(D^3 - D^2 + 4D - 4)x = 8 + 32e^{2t}$$

The roots of the characteristics Eq. are $\pm 2i$ *and* 1

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + X$$

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + Ae^{2t} + B$$

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + 4e^{2t} - 2$$

$$\begin{vmatrix} 2D^2+3D-9 & D^2+7D-14 \\ D+1 & D+2 \end{vmatrix} y = \begin{vmatrix} 2D^2+3D-9 & 4 \\ D+1 & -8e^{2t} \end{vmatrix}$$

$$(D^3-D^2+4D-4)y = -40e^{2t}-4$$

$$y = k_1\cos 2t + k_2\sin 2t + k_3e^t + Y$$

$$y = k_1\cos 2t + k_2\sin 2t + k_3e^t - 5e^{2t} + 1$$

$$We\ must\ have\ 3\ constants,\ used\ 2^{nd}\ Eq.$$

$$(D+1)(c_1\cos 2t + c_2\sin 2t + c_3e^t + 4e^{2t} - 2) +$$

$$(D+2)(k_1\cos 2t + k_2\sin 2t + k_3e^t - 5e^{2t} + 1) = -8e^{2t}$$

$$OR:$$

$$-2c_1\sin 2t + 2c_2\cos 2t + c_3e^t + 8e^{2t} +$$

$$c_1\cos 2t + c_2\sin 2t + c_3e^t + 4e^{2t} - 2 +$$

$$-2k_1\sin 2t + 2k_2\cos 2t + k_3e^t - 10e^{2t} +$$

$$2k_1\cos 2t + 2k_2\sin 2t + 2k_3e^t - 10e^{2t} + 2 = -8e^{2t}$$

$$OR:$$

$$(c_1 + 2c_2 + 2k_1 + 2k_2)\cos 2t +$$

$$(-2c_1 + c_2 - 2k_1 + 2k_2)\sin 2t +$$

$$(2c_3 + 3k_3)\, e^t - 8e^{2t} = -8e^{2t}$$

$$\therefore c_1 + 2c_2 + 2k_1 + 2k_2 = 0$$

$$-2c_1 + c_2 - 2k_1 + 2k_2 = 0$$

$$2c_3 + 3k_3 = 0$$

$$Solve:$$

$$k_1 = \frac{-3c_1 - c_2}{4}, \quad k_2 = \frac{c_1 - 3c_2}{4} \quad and \quad k_3 = -\frac{2}{3}c_3$$

$$\therefore x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^{t} + 4e^{2t} - 2$$

$$y = -\frac{1}{4}(3c_1 + c_2)\cos 2t + \frac{1}{4}(c_1 - 3c_2)\sin 2t$$

$$-\frac{2}{3}c_3 e^{t} - 5e^{2t} + 1$$

Complementary functions and particular integrals for system of equations:

Ex. Solve:

$$\left.\begin{aligned}(D+1)x+(D+2)y+(D+3)z&=-e^{-t}\\(D+2)x+(D+3)y+(2D+3)z&=e^{-t}\\(4D+6)x+(5D+4)y+(20D-12)z&=7e^{-t}\end{aligned}\right]\ldots(1)$$

Solution: As in the case of a single equation make system homogeneous by neglected the terms on the right, getting:

$$\left.\begin{aligned}(D+1)x+(D+2)y+(D+3)z&=0\\(D+2)x+(D+3)y+(2D+3)z&=0\\(4D+6)x+(5D+4)y+(20D-12)z&=0\end{aligned}\right]\ldots(2)$$

Let us now attempt to find solution of this system of the form;

$$x = ae^{mt} \quad , \quad y = be^{mt} \quad , \quad z = ce^{mt} \quad ....(3)$$

Substituting these into the equation (2) *and dividing out the common factor* e^{mx}, *getting;*

$$\left.\begin{array}{l}(m+1)a+(m+2)b+(m+3)c=0 \\ (m+2)a+(m+3)b+(2m+3)c=0 \\ (4m+6)a+(5m+4)b+(20m-12)c=0\end{array}\right]\ldots(4)$$

If nontrivial solution for x, y and z, solutions that do not vanish identically, are to be obtained, it is necessary that a, b and c shall not all be zero.

However, the values $a = b = c = 0$ *satisfy the system* (4) *and in general will be the only solution.*

No other solutions are possible unless the determinate *in* (4) *is equal to zero.*

$$\begin{vmatrix} m+1 & m+2 & m+3 \\ m+2 & m+3 & 2m+3 \\ 4m+6 & 5m+4 & 20m-12 \end{vmatrix} = 0$$

$$Therefor: -(m-1)(m-2)(m-3) = 0$$

$$The\ \ roots\ are: m_1 = 1, m_2 = 2\ and\ m_3 = 3$$

$\therefore$

$$x_1 = a_1 e^{t}, x_2 = a_2 e^{2t}, x_3 = a_3 e^{3t}$$

$$y_1 = b_1 e^{t}, y_2 = b_2 e^{2t}, y_3 = b_3 e^{3t}$$

$$z_1 = c_1 e^{t}, z_2 = c_2 e^{2t}, z_3 = c_3 e^{3t}$$

$For\ m_1 = 1, from (4)$

$2a_1 + 3b_1 + 4c_1 = 0$

$3a_1 + 4b_1 + 5c_1 = 0$

$10a_1 + 9b_1 + 8c_1 = 0$

$These\ Eqs. are\ non\ trivially\ solvable,$

$i.e. has\ a\ solution\ other\ than;$

$a_1 = b_1 = c_1 = 0$

$For\ all\ values\ of\ k_1; a_1 = -k_1, b_1 = 2k_1 and c_1 = -k_1$

$Therefore; x_1 = -k_1 e^t, y_1 = 2k_1 e^t and z_1 = -k_1 e^t$

$$Similarly\ for\ \ m_2 = 2,\ \ from\,(4)$$

$$3a_2 + 4b_2 + 5c_2 = 0$$

$$4a_2 + 5b_2 + 7c_2 = 0$$

$$14a_2 + 14b_2 + 28c_2 = 0$$

$$For\ all\ values\ of\ \ k_2\ ;\ a_2 = 3k_2,\ b_2 = -k_2\ and\ \ c_2 = -k_2$$

$$Therefore,\ x_2 = 3k_2e^{2t},\ y_2 = -k_2e^{2t}\ \ and\ \ z_2 = -k_2e^{2t}$$

$$For\ \ m_3 = 3,$$

$$4a_3 + 5b_3 + 6c_3 = 0$$

$$5a_3 + 6b_3 + 9c_3 = 0$$

$$18a_3 + 19b_3 + 48c_3 = 0$$

$$From\ \ these;$$

$$a_3 = 9k_3,\ b_3 = -6k_3\ and\ c_3 = -k_3$$

$$Therefore,$$

$$x_3 = 9k_3e^{3t},\ y_3 = -6k_3e^{3t}\ \ and\ \ z_3 = -k_3e^{3t}$$

Since Eqs. of Homogeneous (2), are linear;

Sum. of solutions will be also be solutions, So;

$$x = x_1 + x_2 + x_3$$

$$= -k_1 e^{t} + 3k_2 e^{2t} + 9k_3 e^{3t}$$

$$y = y_1 + y_2 + y_3$$

$$= 2k_1 e^{t} - k_2 e^{2t} - 6k_3 e^{3t}$$

$$z = z_1 + z_2 + z_3$$

$$= -k_1 e^{t} - k_2 e^{2t} - k_3 e^{3t}$$

This solution is the complementary function.

For complete solution;

$$X = Ae^{-t}, \quad Y = Be^{-t} \quad and \quad Z = Ce^{-t}$$

Subst. into Eqs. (1), *getting;*

$$(B + 2C)e^{-t} = -e^{-t}$$

$$(A + 2B + C)e^{-t} = e^{-t}$$

$$(2A - B - 32C)e^{-t} = 7e^{-t}$$

Hence; $A = 3, \; B = -1 \; and \; C = 0$

$$X = 3e^{-t}, Y = -e^{-t} \quad and \quad Z = 0$$

$$\therefore \ complete \ \ solution;$$

$$x = -k_1 e^{t} + 3k_2 e^{2t} + 9k_3 e^{3t} + 3e^{-t}$$

$$y = 2k_1 e^{t} - k_2 e^{2t} - 6k_3 e^{3t} - e^{-t}$$

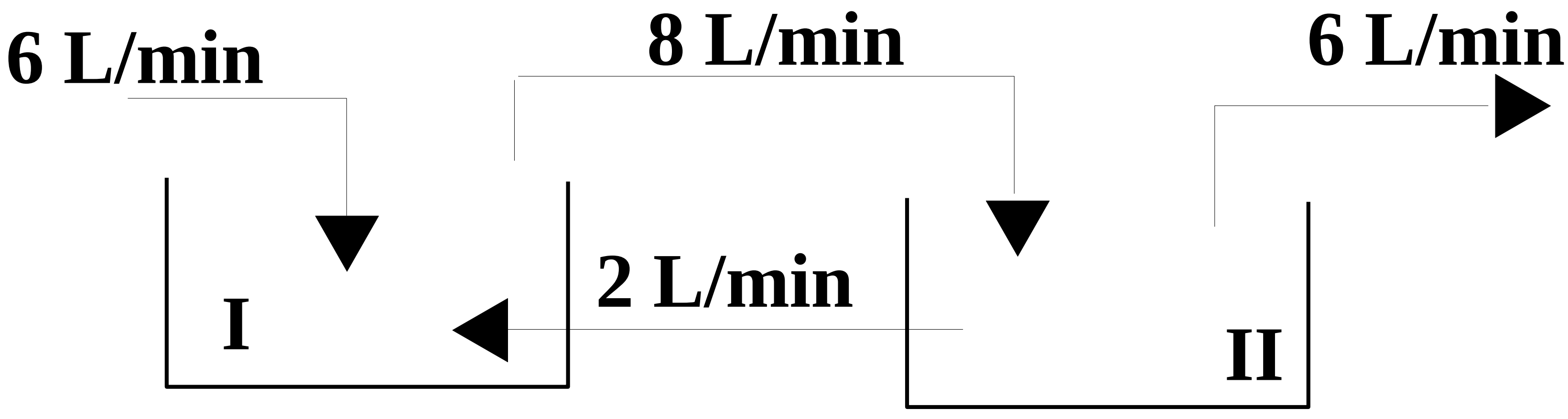
$$z = -k_1 e^{t} - k_2 e^{2t} - k_3 e^{3t}$$

$$Applications$$

$$Modeling \Rightarrow Solving \Rightarrow Interpolation$$

Example:

Two tanks are connected as shown below. The first tank contains initially (100L) of brine containing 50N salt. The second tank contains initially (100L) of brine with 20N salt is dissolved. Starting at t=0, the pumping was applied. If the brine in each tank is kept uniform by stirring. Find the amount of salt in each tank as a function of t.


6 L/min
8 L/min
6 L/min
2 L/min
I
II

$$Solution:$$

$$x : Weight\ of\ \ salt\ \ in\ \ \tan k\ \ I$$

$$y : Weight\ of\ \ salt\ \ in\ \ \tan k\ \ II$$

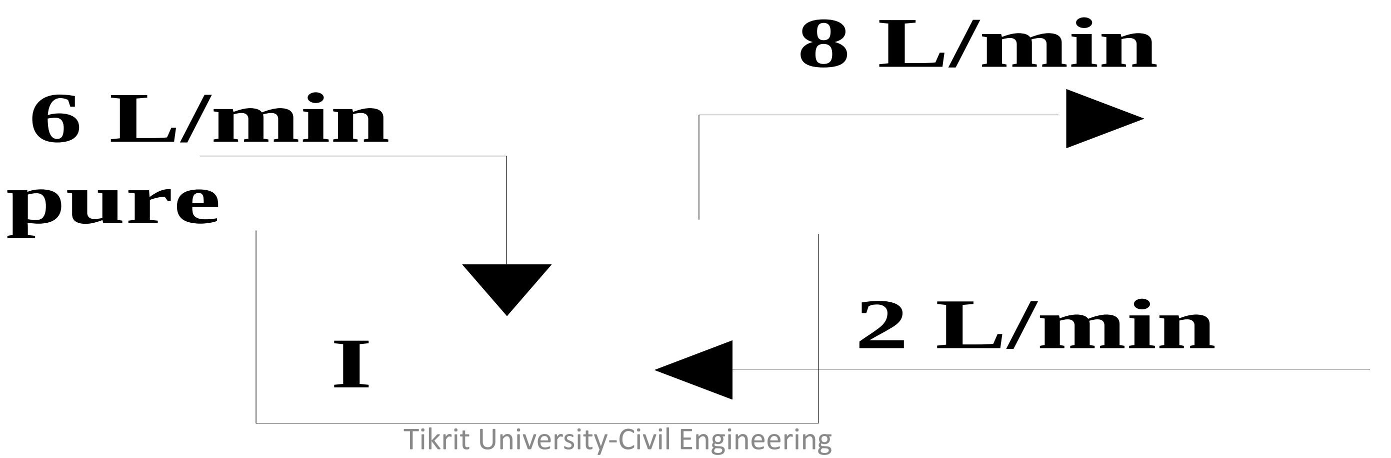
$$\frac{x}{100} = concentration\ of\ \ salt\ \ in\ \tan k\ \ I$$

$$\frac{y}{100} = concentration\ of\ \ salt\ \ in\ \tan k\ \ II$$

$$\frac{dx}{dt} = 2\frac{y}{100} - 8\frac{x}{100}$$

$$\frac{dx}{dt} + 0.08x - 0.02y = 0$$

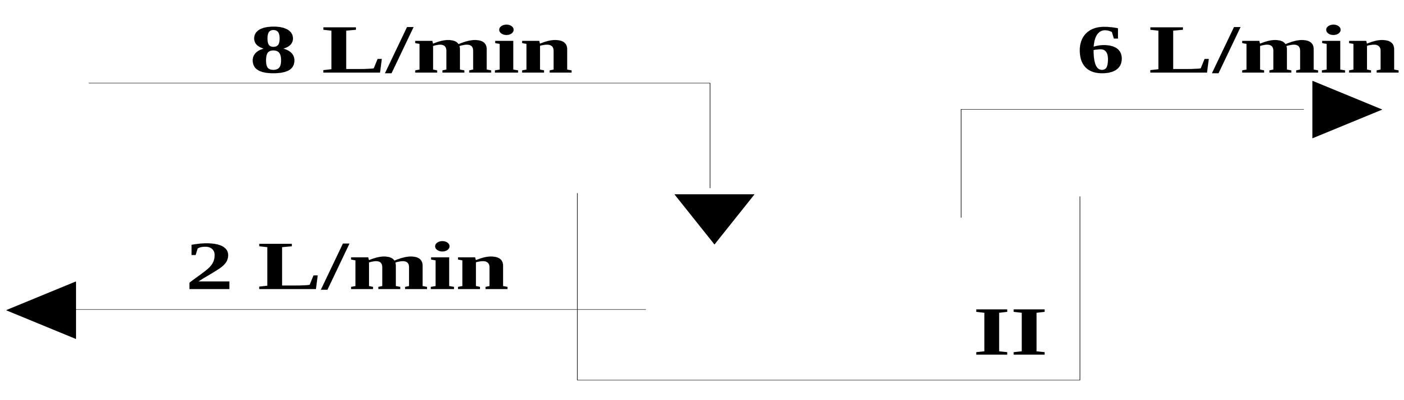
$$(D + 0.08)x - 0.02y = 0 \qquad \ldots\ldots\ldots\ldots(1)$$



$$\frac{dy}{dt} = 8\frac{x}{100} - 2\frac{y}{100} - 6\frac{y}{100}$$

$$\frac{dy}{dt} = 0.08x - 0.08y$$

$$0.08x - (D + 0.08)y = 0 \ldots\ldots\ldots\ldots(2)$$



$$\begin{bmatrix} D+0.08 & -0.02 \\ 0.08 & -(D+0.08) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 0 & -0.02 \\ 0 & -(D+0.08) \end{vmatrix}}{\begin{vmatrix} D+0.08 & -0.02 \\ 0.08 & -(D+0.08) \end{vmatrix}}$$

$$[\{(D+0.08)(-(D+0.08))\} - (0.08(-0.02))]x = 0$$

$$[-(D^2 + 0.16D + 0.0064) + 0.0016]x = 0$$

$$[-(D^2+0.16D+0.0064)+0.0016]x=0$$

$$(-D^2-0.16D-0.0048)x=0$$

$$(D^2+0.16D+0.0048)x=0$$

$$(D+0.12)(D+0.04)x=0$$

$$The\ \ roots\ \ are;\ \ -0.12\ \ and\ \ -0.04$$

$$\therefore x=c_1e^{-0.12t}+c_2e^{-0.04t}$$

$$Also;\ \ y=c_3e^{-0.12t}+c_4e^{-0.04t}$$

$$x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y = c_3 e^{-0.12t} + c_4 e^{-0.04t}$$

$$Subst.\ \mathrm{int}\,o\ Eq.\,(1)$$

$$(D + 0.08)x - 0.02y = 0$$

$$(-0.12c_1 e^{-0.12t} - 0.04c_2 e^{-0.04t}) +$$

$$0.08(c_1 e^{-0.12t} + c_2 e^{-0.04t}) -$$

$$0.02(c_3 e^{-0.12t} + c_4 e^{-0.04t}) = 0$$

$$By\ equating\ the\ coefficients:$$

$$-0.12c_1 + 0.08c_1 - 0.02c_3 = 0$$

$$-0.04c_2 + 0.08c_2 - 0.02c_4 = 0$$

$$\therefore c_3 = \frac{-0.04}{0.02}c_1 = -2c_1$$

$$c_4 = \frac{0.04}{0.02}c_2 = 2c_2$$

$$\therefore x = c_1e^{-0.12t} + c_2e^{-0.04t}$$

$$y = -2c_1e^{-0.12t} + 2c_2e^{-0.04t}$$

$$x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y = -2c_1 e^{-0.12t} + 2c_2 e^{-0.04t}$$

$$At \;\; t = 0, \; x = 50 \; and \;\; y = 20$$

$$50 = c_1 + c_2$$

$$20 = -2c_1 + 2c_2$$

$$c_1 = 20 \quad and \quad c_2 = 30$$

$$\therefore x = 20e^{-0.12t} + 30e^{-0.04t}$$

$$y = -40e^{-0.12t} + 60e^{-0.04t}$$

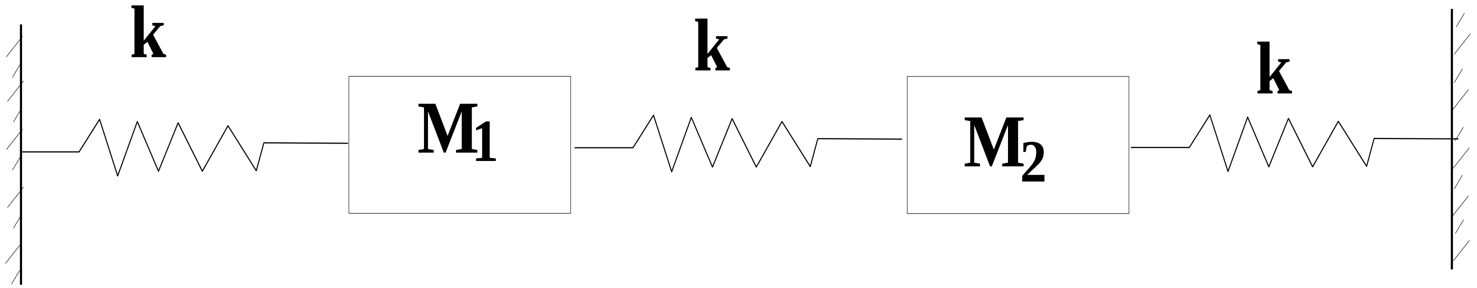
Mechanical vibration

Example:

The system with two degree of freedom beings to move under the following condition;

Initial velocity $\dot{x}_1(0) = \sqrt{3k}$ and $\dot{x}_2(0) = \sqrt{3k}$

Neglected the friction, derive the D.E. covering free vibration of system show below:

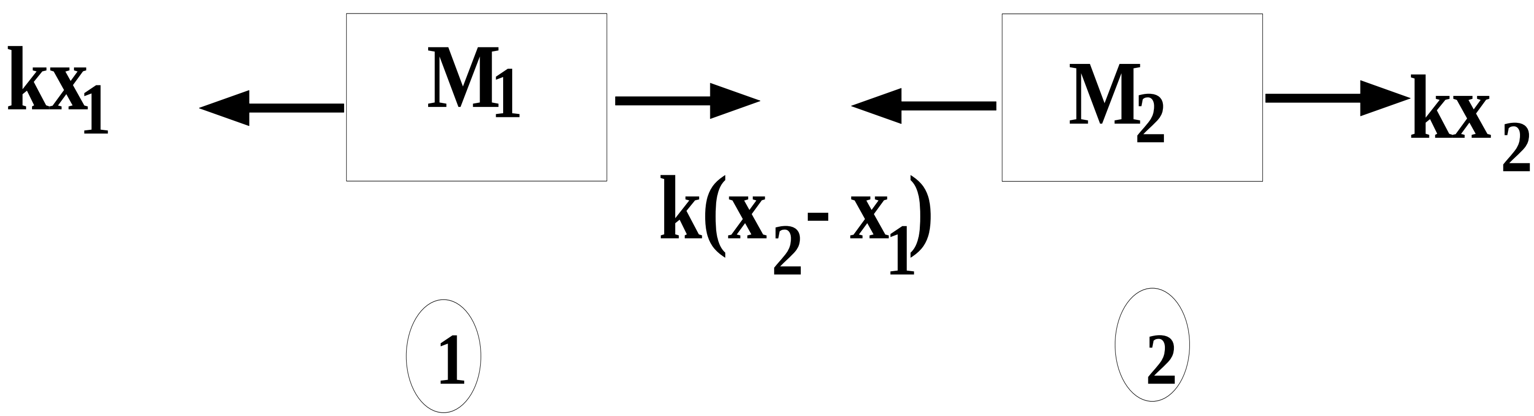


$M_1 = M_2 = 1$ **k= spring constant**

Solution:

$$Mass * acceleration = \sum force\ in\ the\ direction\ of\ motion$$

$$Force = spring\ constant * displacement = k * x$$



$$From\ (1)$$

$$Mass * acceleration = \sum fx$$

$$M_1 \ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

$$M_1 = 1$$

$$\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$(D^2 + 2k)x_1 - kx_2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$From\ (2)$$

$$M_2\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$M_1 = 1$$

$$-kx_1 + (D^2 + 2k)x_2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\begin{bmatrix} D^2 + 2k & -k \\ -k & D^2 + 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(D^4 + 4kD^2 + 4k^2 - k^2)x_1 = 0$$

$$(m^2 + k)(m^2 + 3k) = 0$$

$$m = \pm i\sqrt{k} and \pm i\sqrt{3k}$$

$$x_1 = c_1 \cos\sqrt{k}\, t + c_2 \sin\sqrt{k}\, t + c_3 \cos\sqrt{3k}\, t + c_4 \sin\sqrt{3k}\, t$$

$$x_2 = c_5 \cos\sqrt{k}\, t + c_6 \sin\sqrt{k}\, t + c_7 \cos\sqrt{3k}\, t + c_8 \sin\sqrt{3k}\, t$$

$$x_1 = c_1 \cos\sqrt{k}\,t + c_2 \sin\sqrt{k}\,t + c_3 \cos\sqrt{3k}\,t + c_4 \cos\sqrt{3k}\,t$$

$$x_2 = c_5 \cos\sqrt{k}\,t + c_6 \sin\sqrt{k}\,t + c_7 \cos\sqrt{3k}\,t + c_8 \cos\sqrt{3k}\,t$$

$Subst.\, x_1(t) and x_2(t)\ \text{int}\, o\ Eq. \rightleftarrows\rightleftarrows\ (1)$

$$c_5 = c_1, c_6 = c_2, c_7 = -c_3 and c_8 = -c_4$$

$$\therefore x_1(t) = c_1 \cos\sqrt{k}\,t + c_2 \sin\sqrt{k}\,t + c_3 \cos\sqrt{3k}\,t + c_4 \sin\sqrt{3k}\,t$$

$$x_2(t) = c_1 \cos\sqrt{k}\,t + c_2 \sin\sqrt{k}\,t - c_3 \cos\sqrt{3k}\,t - c_4 \sin\sqrt{3k}\,t$$